

ON p -ADIC POWER SERIES

Branko DRAGOVICH, Institute of Physics, P.O.Box 57, 11001 Belgrade, Yugoslavia; dragovich@phy.bg.ac.yu

Abstract We obtained the region of convergence and the summation formula for some modified generalized hypergeometric series (1.2). We also investigated rationality of the sums of the power series (1.3). As a result the series (1.4) cannot be the same rational number in all \mathbb{Z}_p .

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1 Introduction

We are interested in investigation of various properties of some p -adic power series of the form

$$\sum_{n=0}^{\infty} A_n x^n, \quad (1.1)$$

where coefficients $A_n \in \mathbb{Q}$ and variable $x \in \mathbb{Q}_p$. Such series are often encountered in p -adic analysis [1] as well as in its applications in mathematical and theoretical physics (for a review, see, e.g. Refs. 2-5). Due to rationality of A_n , the series (1.1) can be simultaneously considered in all \mathbb{Q}_p and in \mathbb{R} . It is of particular interest to find all rational points for some classes of the series (1.1). Some previous author's investigations on p -adic series of the form (1.1) were presented at the Fourth International Conference on p -Adic Analysis ([6] and references therein).

In this contribution we mainly consider some general properties of the

series

$$\sum_{n=0}^{\infty} a_n R_{k,l}(n) x^n , \quad (1.2)$$

where a_n are coefficients of the generalized hypergeometric series and $R_{k,l}(n) = P_k(n)/Q_l(n)$ are rational functions in $n \in \mathbb{Z}_0 = \{0, 1, 2, \dots\}$. We also examine in some details the series

$$\sum_{n=0}^{\infty} n! P_k(n) x^n , \quad (1.3)$$

where $P_k(n)$ is a polynomial of degree k . In particular, we show that

$$\sum_{n=0}^{\infty} n! \quad (1.4)$$

cannot be the same rational number in \mathbb{Z}_p for every p .

Note that in virtue of non-archimedean properties of p -adic norm, the necessary condition is also the sufficient one for the series (1.1) to be convergent, *i.e.* (1.1) is p -adic convergent for some x iff

$$|A_n x^n|_p \rightarrow 0 , \quad n \rightarrow \infty . \quad (1.5)$$

It is worth mentioning that Schikhof's book [1] contains an excellent introductory course to analysis of p -adic series and, if necessary, can be used to better understand some of our considerations.

2 Generalized Hypergeometric Series

Let $P_k(n)$ be a polynomial

$$P_k(n) = C_k n^k + C_{k-1} n^{k-1} + \dots + C_0 , \quad 0 \neq C_k, C_{k-1}, \dots, C_0 \in \mathbb{Q} , \quad (2.1)$$

in $n \in \mathbb{Z}_0$ of degree k . Let also $Q_l(n)$ be another polynomial of degree l ,

$$Q_l(n) = D_l n^l + D_{l-1} n^{l-1} + \dots + D_0 , \quad 0 \neq D_l, D_{l-1}, \dots, D_0 \in \mathbb{Q} , \quad (2.2)$$

with restriction $Q_l(n) \neq 0$ for every $n \in \mathbb{Z}_0$.

We will call the R -modified generalized hypergeometric series

$$\begin{aligned} & {}_r F_s(\alpha_1, \alpha_2, \dots, \alpha_r; \beta_1, \beta_2, \dots, \beta_s; R_{k,l}; x) \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \dots (\alpha_r)_n}{(\beta_1)_n (\beta_2)_n \dots (\beta_s)_n} R_{k,l}(n) \frac{x^n}{n!} , \end{aligned} \quad (2.3)$$

where $R_{k,l}(n)$ is a rational function

$$R_{k,l}(n) = \frac{P_k(n)}{Q_l(n)} \quad (2.4)$$

with polynomials $P_k(n)$ and $Q_l(n)$ defined by (2.1) and (2.2), respectively, and $(u)_0 = 1$, $(u)_n = u(u+1) \cdots (u+n-1)$ for $n \geq 1$. When $R_{k,l}(n) \equiv 1$ one gets the standard definition of the generalized hypergeometric series.

Proposition 1 The R -modified hypergeometric series defined by (2.3), where $\alpha_1, \alpha_2, \dots, \alpha_r \in \mathbb{Z}_+ = \{1, 2, 3, \dots\}$ and $\beta_1, \beta_2, \dots, \beta_s \in \mathbb{Z}_+$, is p -adically convergent in the region

$$|x|_p < p^{\frac{r-s-1}{p-1}}. \quad (2.5)$$

Proof: Note that

$$(u)_n = \frac{(u+n-1)!}{(u-1)!} \quad (2.6)$$

if $u \in \mathbb{Z}_+$. Then p -adic norm of the general term in (2.3) can be written as

$$\begin{aligned} & \left| \frac{(\beta_1-1)! \cdots (\beta_s-1)!}{(\alpha_1-1)! \cdots (\alpha_r-1)!} \right|_p \left| \frac{(\alpha_1+n-1)! \cdots (\alpha_r+n-1)!}{(\beta_1+n-1)! \cdots (\beta_s+n-1)!} \right|_p \\ & \times \left| \frac{P_k(n)}{Q_l(n)} \right|_p \left| \frac{x^n}{n!} \right|_p. \end{aligned} \quad (2.7)$$

Recall that

$$|m!|_p = p^{-\frac{m-\sigma_m}{p-1}}, \quad m \in \mathbb{Z}_+, \quad (2.8)$$

where σ_m is the sum of digits in the expansion of m over the base p . Since the first factor does not depend on n and $|P_k(n)|_p/|Q_l(n)|_p$ is bounded it suffices to analyse

$$\left| \frac{(\alpha_1+n-1)! \cdots (\alpha_r+n-1)!}{(\beta_1+n-1)! \cdots (\beta_s+n-1)!} \right|_p \left| \frac{x^n}{n!} \right|_p. \quad (2.9)$$

For large enough n (2.9) behaves like

$$\left(p^{-\frac{r-s-1}{p-1}} |x|_p \right)^n, \quad (2.10)$$

which tends to zero as $n \rightarrow \infty$ if

$$p^{-\frac{r-s-1}{p-1}} |x|_p < 1, \quad (2.11)$$

what just gives (2.5).

Note that (2.5) does not depend on the values of the parameters $\alpha_1, \alpha_2, \dots, \alpha_r$ and $\beta_1, \beta_2, \dots, \beta_s$ but only on their multiplicity r and s . For the Gauss series

$${}_2F_1(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} x^n \quad (2.12)$$

one obtains $|x|_p < 1$, like $|x|_{\infty} < 1$ in the real case.

Let us now turn to finding the corresponding summation formula.

Proposition 2 Let ${}_rF_s(\alpha_1, \alpha_2, \dots, \alpha_r; \beta_1, \beta_2, \dots, \beta_s; R_{k,l}; x)$ be an R -modified generalized hypergeometric series defined by (2.3) with the region of convergence given by (2.5). Then the following summation formula

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_r)_n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_s)_n n!} \left[\frac{(\alpha_1 + n)(\alpha_2 + n) \cdots (\alpha_r + n)}{(\beta_1 + n)(\beta_2 + n) \cdots (\beta_s + n)(n + 1)} \right. \\ & \times \left. \frac{A_{\mu}(n + 1)}{B_{\nu}(n + 1)} x - \frac{A_{\mu}(n)}{B_{\nu}(n)} \right] x^n = - \frac{A_{\mu}(0)}{B_{\nu}(0)} \end{aligned} \quad (2.13)$$

is valid, where $A_{\mu}(n)$ and $B_{\nu}(n)$ are polynomials in $n \in \mathbb{Z}_0$ of the form (2.1) and (2.2), respectively.

Proof: The left hand side of (2.13) can be rewritten in the form

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_r)_n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_s)_n} \frac{A_{\mu}(n)}{n! B_{\nu}(n)} x^n \\ & - \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_r)_n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_s)_n} \frac{A_{\mu}(n)}{n! B_{\nu}(n)} x^n \end{aligned}$$

which, by mutual cancellation of all terms except term for $n = 0$, gives just $-\frac{A_{\mu}(0)}{B_{\nu}(0)}$.

Although based on a simple derivation, (2.13) leads to the rather non-trivial results. Notice that always when

$$R_{k,l}(n) = \frac{(\alpha_1 + n)(\alpha_2 + n) \cdots (\alpha_r + n)}{(\beta_1 + n)(\beta_2 + n) \cdots (\beta_s + n)} \frac{t}{n + 1} \frac{A_{\mu}(n + 1)}{B_{\nu}(n + 1)} - \frac{A_{\mu}(n)}{B_{\nu}(n)}, \quad (2.14)$$

where $A_{\mu}(n)$ and $B_{\nu}(n)$ are arbitrary polynomials defined like (2.1) and (2.2), respectively, if $x = t$ we have the resulting rational sum of (2.3), which does not depend on x and is equal to $-A_{\mu}(0)/B_{\nu}(0)$. Of course, the parameter t and the argument x belong to the region of convergence (2.5).

A generalized hypergeometric series is defined by its parameters, $\alpha_1, \alpha_2, \dots, \alpha_r$ and $\beta_1, \beta_2, \dots, \beta_s$. For a given generalized hypergeometric series there are many possibilities to choose rational functions $R_{k,l}(n)$ (2.14)

with the corresponding rational sums $-A_\mu(0)/B_\nu(0)$. Let us notice some characteristic cases with $B_\nu(n) \equiv 1$. $A_\mu(n)$ may contain any partial or complete product of factors in the denominator: $\beta_1 + n - 1, \beta_2 + n - 1, \dots, \beta_s + n - 1, n$. In the case when $A_\mu(n)$ includes n as a factor then $A_\mu(0) = 0$ and the sum of the corresponding series (2.13) will be also equal to zero. An extreme case is

$$A_\mu(n) = (\beta_1 + n - 1)(\beta_2 + n - 1) \cdots (\beta_s + n - 1)nB_\nu(n) \quad (2.15)$$

that gives in (2.14) the polynomial

$$P_k(n) = (\alpha_1 + n)(\alpha_2 + n) \cdots (\alpha_r + n)tA_\mu(n + 1) - A_\mu(n) \quad (2.16)$$

instead of a rational function $R_{k,l}(n)$. Thus we have

$$\sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \cdots (\alpha_r)_n}{(\beta_1)_n(\beta_2)_n \cdots (\beta_s)_n} P_k(n) \frac{t^n}{n!} = 0 \quad (2.17)$$

if $P_k(n)$ has the form (2.16).

3 Series $\sum_{n=0}^{\infty} n! P_k(n) x^n$

This series can be regarded as a simple example of the R -modified generalized hypergeometric series, *i.e.*

$${}_2F_0(1, 1; P_k; x) = \sum_{n=0}^{\infty} n! P_k(n) x^n. \quad (3.1)$$

Because of its relative simplicity the series (3.1) is suitable for examination of various p -adic properties. Power series (3.1) is divergent in the real case. From (2.5) it follows that its p -adic region of convergence is $|x|_p < p^{1/(p-1)}$ and it yields in \mathbb{Q}_p :

$$x \in \mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}. \quad (3.2)$$

As a consequence of (3.2) we may take for x any integer and the series (3.1) will be p -adically convergent for every prime p .

The corresponding summation formula is

$$\sum_{n=0}^{\infty} n! [(n+1)A_{k-1}(n+1)x - A_{k-1}(n)]x^n = -A_{k-1}(0). \quad (3.3)$$

For $x = 1$ it can be rewritten in the more suitable form

$$\sum_{n=0}^{\infty} n!(n^k + u_k) = v_k , \quad (3.4)$$

where $(n+1)A_{k-1}(n+1) - A_{k-1}(n) = n^k + u_k$, $v_k = -A_{k-1}(0)$. One can easily see that $u_k = A_{k-1}(1) - A_{k-1}(0)$.

It is very useful to have expressions for finite (partial) sums of (3.4).

Proposition 3 If

$$S_n^{(k)} = \sum_{i=0}^{n-1} i!i^k , \quad k \in \mathbb{Z}_0 , \quad (3.5)$$

then

$$S_n^{(k+1)} = -\delta_{0k} - kS_n^{(k)} - \sum_{l=0}^{k-1} \binom{k+1}{l} S_n^{(l)} + n!n^k \quad (3.6)$$

is a recurrent relation, where δ_{0k} is the Kronecker symbol ($\delta_{0k} = 1$ if $k = 0$ and $\delta_{0k} = 0$ if $k \neq 0$).

Proof:

$$\begin{aligned} S_n^{(k)} &= \delta_{0k} + \sum_{i=0}^{n-2} (i+1)!(i+1)^k = \delta_{0k} + \sum_{i=0}^{n-1} i!(i+1)^{k+1} - n!n^k \\ &= \delta_{0k} + \sum_{l=0}^{k+1} \binom{k+1}{l} S_n^{(l)} - n!n^k . \end{aligned}$$

Applying successively the recurrent relation (3.6) we obtain summation formula of the form

$$\sum_{i=0}^{n-1} i!(i^k + u_k) = v_k + n!A_{k-1}(n) , \quad (3.7)$$

where $A_{k-1}(n)$ is a polynomial of degree $k-1$ in n with integer coefficients. As an illustration, here are the first four examples:

$$\begin{aligned} \text{(a)} \quad & \sum_{i=0}^{n-1} i!i = -1 + n! , \\ \text{(b)} \quad & \sum_{i=0}^{n-1} i!(i^2 + 1) = 1 + n!(n-1) , \\ \text{(c)} \quad & \sum_{i=0}^{n-1} i!(i^3 - 1) = 1 + n!(n^2 - 2n - 1) , \\ \text{(c)} \quad & \sum_{i=0}^{n-1} i!(i^4 - 2) = -5 + n!(n^3 - 3n^2 + 5) . \end{aligned} \quad (3.8)$$

Table 1 contains the first eleven values of u_k and v_k .

k	1	2	3	4	5	6	7	8	9	10	11
u_k	0	1	-1	-2	9	-9	-50	267	-413	-2180	17731
v_k	-1	1	1	-5	5	21	-105	141	777	-5513	13209

In a similar way to the Proposition 3 one can obtain recurrent relations for u_k and v_k :

$$\begin{aligned} u_{k+1} &= -ku_k - \sum_{l=1}^{k-1} \binom{k+1}{l} u_l + 1, \quad u_1 = 0, \quad k \geq 1, \\ v_{k+1} &= -kv_k - \sum_{l=1}^{k-1} \binom{k+1}{l} v_l - \delta_{0k}, \quad k \geq 0. \end{aligned} \quad (3.9)$$

It is worth noting that $i^k + u_k$ in (3.7) is a simplified form of $P_k(i)$ which gives rational sum of (3.1) if $x = 1$. Such $P_k(i) = i^k + u_k$ are suitable to obtain a general expression for the series (3.1) with rational sum at $x = 1$. In fact, the generalized form of (3.7) is

$$\sum_{i=0}^{n-1} i! P_k(i) = V_k + n! B_{k-1}(n), \quad k \geq 1, \quad (3.10)$$

where $P_k(i) = \sum_{r=0}^k C_r i^r$ with $C_0 = \sum_{r=1}^k C_r u_r$, $V_k = \sum_{r=1}^k C_r v_r$, $B_{k-1}(n) = \sum_{r=1}^k C_r A_{r-1}(n)$ and $C_1, C_2, \dots, C_k \in \mathbb{Q}$.

The above consideration performed for $x = 1$ can be extended to other positive integers x with some other values of u_k and v_k .

Let us turn now to the sum of the power series (3.1) and investigate some of its rationality problem at $x \in \mathbb{Z}_+$. It is useful to start with the simplest case, *i.e.* $P_k(n) \equiv 1$.

Theorem 1 Let x be a given positive integer. If p -adic sum of the power series

$$\sum_{n=0}^{\infty} n! x^n \quad (3.11)$$

is a rational number then it cannot be the same in \mathbb{Z}_p for every p .

Proof: Suppose there is such $x = t \in \mathbb{Z}_+$ that there exists p -adic rational sum

$$\sum_{n=0}^{\infty} n! t^n = \frac{a(t)}{b(t)} \quad a(t) \in \mathbb{Z}, \quad b(t) \in \mathbb{Z}_+ \quad (3.12)$$

the same for every p . Let $S_n(t)$ be

$$S_n(t) = \sum_{i=0}^{n-1} i!t^i . \quad (3.13)$$

Since $i!t^i < (n-1)!t^{n-1}$ when $0 \leq i \leq n-2$ one has $S_n(t) = 0! + 1!t + 2!t^2 + \dots + (n-2)!t^{n-2} + (n-1)!t^{n-1} < (n-1)!(n-1)t^{n-1} + (n-1)!t^{n-1} = n!t^{n-1} \leq n!t^n$. Thus we have inequality

$$0 < S_n < n!t^n , \quad n > 2, \quad t \geq 1 . \quad (3.14)$$

For a fixed $b(t) \in \mathbb{Z}_+$ one can write $b(t)S_n(t) = b(t)[0! + 1!t + 2!t^2 + \dots + (n-2)!t^{n-2} + (n-1)!t^{n-1}] < b(t)[(n-1)!t^{n-1} + (n-1)!t^{n-1}] = 2b(t)(n-1)!t^{n-1} < n!t^n$ if $2b(t) < nt$, *i.e.*

$$0 < b(t)S_n(t) < n!t^n , \quad 2b(t) < nt . \quad (3.15)$$

According to (3.12) one has $a(t) = b(t)S_n(t) + n!t^n b(t)[1 + (n+1)t + \dots]$. Due to our assumption, $b(t)[1 + (n+1)t + \dots]$ must be the same rational integer in all \mathbb{Z}_p and we get congruence

$$a(t) \equiv b(t)S_n(t) \pmod{n!t^n} , \quad n \geq 0, \quad t \geq 1 . \quad (3.16)$$

The value of $a(t)$ belongs to the one of the following three possibilities: (i) $a(t) > 0$, (ii) $a(t) < 0$ and (iii) $a(t) = 0$. Consider each of these possibilities. According to (3.15) and (3.16) for large enough n we have:

$$\begin{aligned} (i) \quad & 0 < a(t) < n!t^n , \\ & 0 < b(t)S_n(t) < n!t^n , \\ & a(t) \equiv b(t)S_n(t) \pmod{n!t^n} ; \\ (ii) \quad & -n!t^n < a(t) < 0 , \\ & 0 < b(t)S_n(t) < n!t^n , \\ & a(t) \equiv b(t)S_n(t) \pmod{n!t^n} ; \\ (iii) \quad & a(t) = 0 , \\ & 0 < b(t)S_n(t) < n!t^n , \\ & a(t) \equiv b(t)S_n(t) \pmod{n!t^n} . \end{aligned} \quad (3.17)$$

Analysing the conditions in (3.17) we find the following candidates for solution:

$$\begin{aligned} (i) \quad & a(t) = b(t)S_n(t) , \\ (ii) \quad & a(t) = b(t)S_n(t) - n!t^n \end{aligned} \quad (3.18)$$

and (iii) without solution. Since $a(t)$ must be a fixed integer we conclude that the solutions (3.18), which depend on n , are impossible.

As a particular case of the Theorem 1 we have that the sum of the series (1.4) cannot be the same rational number in all \mathbb{Z}_p . Note an earlier assertion (see [1], p.17) that $\sum_{n=0}^{\infty} n!$ cannot be rational in \mathbb{Z}_n for every n .

Theorem 2 For fixed k and x the sum of the power series

$$\sum_{n=0}^{\infty} n!n^k x^n, \quad k \in \mathbb{Z}_0, \quad x \in \mathbb{Z}_+ \setminus \{1\}, \quad (3.19)$$

cannot be the same rational number in \mathbb{Z}_p for every p .

Proof: When $k = 0$ it follows from Theorem 1. Dividing (3.3) by x , for $k \geq 1$ one has

$$\sum_{n=0}^{\infty} n![n^k + u_k(x)]x^n = v_k(x), \quad x \in \mathbb{Z}_p \setminus \{0\}, \quad (3.20)$$

as a generalization of (3.4). Analysing the system of linear equations for coefficients of the polynomial $A_{k-1}(n)$, which follows from

$$(n+1)A_{k-1}(n+1) - \frac{A_{k-1}(n)}{x} = n^k + u_k(x), \quad (3.21)$$

we conclude that $u_k(x)$ has the form

$$u_k(x) = \frac{-1 + xF_{k-1}(x)}{x^k}, \quad (3.22)$$

where $F_{k-1}(x)$ is a polynomial in x of degree $k-1$ with integer coefficients (for $k = 1, \dots, 4$ see the Table 2). The series (3.19) might be the same rational number in all \mathbb{Z}_p for some $x \in \mathbb{Z}_+$ iff

$$-1 + xF_{k-1}(x) = 0. \quad (3.23)$$

However $F_{k-1}(x)$ is a polynomial with integer coefficients and eq. (3.23) has no solutions in $x \in \mathbb{Z}_+ \setminus \{1\}$.

Among the series of the form

$$\sum_{n=0}^{\infty} n!n^k, \quad k \in \mathbb{Z}_+, \quad (3.24)$$

it is easy to see (Table 1 and (3.8)) that

$$\sum_{n=0}^{\infty} n!n = -1 \quad (3.25)$$

Table 2 Expressions for $u_k(x)$ and $v_k(x)$ ($k = 1, \dots, 4$) illustrate some of our conclusions.

k	1	2	3	4
$u_k(x)$	$\frac{-1+x}{x}$	$\frac{-1+3x-x^2}{x^2}$	$\frac{-1+6x-7x^2+x^3}{x^3}$	$\frac{-1+10x-25x^2+15x^3-x^4}{x^4}$
$v_k(x)$	$-\frac{1}{x}$	$\frac{-1+2x}{x^2}$	$\frac{-1+5x-3x^2}{x^3}$	$\frac{-1+9x-17x^2+4x^3}{x^4}$

in \mathbb{Z}_p for every p . According to the Table 1 the sum of the series

$$\sum_{n=0}^{\infty} n!n^k, \quad k = 2, 3, \dots, 11, \quad (3.26)$$

cannot be the same rational number (for a fixed k) in all \mathbb{Z}_p .

Proposition 4 The sum of the series

$$\sum_{n=0}^{\infty} n!n^{q+1}, \quad q = \text{any of prime numbers}, \quad (3.27)$$

cannot be the same rational number (for a fixed q) in \mathbb{Z}_p for every prime p .

Proof: According to the recurrent relations (3.9) one has for any prime number q that $u_{q+1} \equiv 1 \pmod{q}$ and $v_{q+1} \equiv 1 \pmod{q}$. Thus, $u_{q+1} \neq 0$ and v_{q+1} is a rational integer.

It is unlikely that $\sum_{n=0}^{\infty} n!n^k$ is a rational number if $k \neq 1$. Thus there is a sense to introduce the following

Conjecture The sum of the series

$$\sum_{n=0}^{\infty} n!n^k, \quad k \in \mathbb{Z}_0$$

is a rational number in all \mathbb{Z}_p iff $k = 1$. Or, in the more general form, p -adic sum of the power series

$$\sum_{n=0}^{\infty} n!n^k x^n, \quad k \in \mathbb{Z}_0, \quad x \in \mathbb{Z}_+$$

is a rational number iff $k = x = 1$.

4 Concluding Remarks

It is worth noting that the p -adic power series

$$\sum_{j=0}^{\infty} (n+1)_j x^j, \quad x \in \mathbb{Z}_+$$

cannot be a rational integer in any \mathbb{Z}_p as well as the same rational number in all \mathbb{Z}_p . This follows from identity

$$\sum_{n=0}^{\infty} n! x^n = S_n(x) + n! x^n \sum_{j=0}^{\infty} (n+1)_j x^j$$

and the proof of the Theorem 1.

It is clear that the p -adic hypergeometric series (2.12) satisfies the corresponding hypergeometric differential equation, i.e.

$$x(1-x)w'' + [\gamma - (\alpha + \beta + 1)x]w' - \alpha\beta w = 0,$$

where $w = {}_2F_1(\alpha, \beta; \gamma; x)$. Let us also notice that the p -adic series

$$F_\nu(x) = \sum_{n=0}^{\infty} n! x^{n+\nu}, \quad \nu \in \mathbb{Z}_+,$$

is a solution of the following differential equation

$$\left(\frac{d^\nu}{dx^\nu} - \frac{1}{x^{2\nu}}\right)F_\nu(x) = f_\nu(x), \quad x \in \mathbb{Z}_p \setminus \{0\},$$

where

$$f_\nu(x) = - \sum_{l=0}^{\nu-1} \frac{l!}{x^{\nu-l}}.$$

The series

$$F(x) = \sum_{n=0}^{\infty} n! x^n$$

may be regarded as an analytic solution of the differential equation

$$x^2 F''(x) + (3x - 1)F'(x) + F(x) = 0.$$

Many of the above results, obtained for $x \in \mathbb{Z}_+$, may be extended to $x \in \mathbb{Z} \setminus \{0\}$ and it will be done elsewhere.

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